Effect of angular momentum conservation in the phase transitions of collapsing systems

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The effect of angular momentum conservation within microcanonical thermodynamics is considered. This is relevant in self-gravitating systems, where angular momentum is conserved and the collapsing nature of the forces makes the microcanonical ensemble the proper statistical description of the physical processes. The microcanonical distribution function with nonvanishing angular momentum is obtained as a function of the coordinates of the particles. As an example, a simple model, introduced by Thirring long ago [Z. Phys. **235**, 339 (1970)], is worked out. The phase diagram contains three phases: For low values of the angular momentum L the system behaves as the original model, showing a complete collapse at low energies and a convex intruder in the entropy. For intermediate values of L the collapse at low energies is not complete, and the entropy still has a convex intruder. For large L there is neither collapse nor anomalies in the thermodynamical quantities. A short discussion of the extension of these results to more realistic situations is given.

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I. INTRODUCTION

Conventional thermodynamics applies to systems whose forces have saturation property, i.e., the minimum of the potential energy of large systems is proportional to the number of particles. If this is the case, macroscopic parts have negligible interactions and then the macroscopically conserved quantities are extensive, the thermodynamic potentials are homogeneous functions of these quantities, and large systems can be studied in the thermodynamical limit. These properties hold if the so called stability condition is verified. For a system of N classical particles interacting via a two body potential $\phi(r)$ the stability condition states that there must exist a positive constant E_0 such that, for each configuration $\{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\}$, the following inequality is obeyed [1]:

$$\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)=\tfrac{1}{2}\sum_{i\neq j}\phi(|\boldsymbol{r}_i-\boldsymbol{r}_j|)\geqslant -NE_0. \qquad (1)$$

This condition is realized in very specific situations. For instance, Eq. (1) holds for potentials repulsive enough at short distances, and decaying faster than $1/r^D$ at long distances (D is the dimensionality of the space). The Lennard-Jones potential is an example. Systems with purely attractive potentials are always unstable, even if the forces are short ranged, such as those studied in Refs. [2,3]. Owing to the long range of the potential, self-gravitating systems are unstable, even if the short distance singularity is removed by considering particles endowed with a hard core. When the stability condition does not hold, the system undergoes a phase transition from a high energy homogeneous phase (HP) to a collapsing phase (CP) at low energies [4–8]. Recently, it has been discovered [7,8] that there is a dynamical characterization of the two phases: the single-particle motion is superdiffusive in the CP, and ballistic in the HP. The low energy regime is not a proper thermodynamical phase, since it is not homogeneous. Nevertheless, we shall use the term ''collapsing phase'' throughout this paper.

It is well known that, as a consequence of the virial theorem, self-gravitating systems have a negative specific heat [9,10]. In fact, the common feature of all these systems seems to be the appearance of an interval of energies where the microcanonical specific heat is negative [4,5,7]. This is the transition region between the HP and CP.

A negative specific heat may also be observed in systems with a finite number of degrees of freedom (see Ref. [11]), but, for stable systems, van Hove's theorem [12] states that, in the thermodynamical limit, the entropy at fixed volume is a concave function of the energy. As a consequence, the specific heat cannot be negative. However, the thermodynamical limit does not exist for unstable systems, and van Hove's theorem does not apply. Indeed, as mentioned above, it has been observed that the microcanonical specific heat remains negative in an energy interval after increasing the number of particles [4–8]. Since the canonical specific heat is always positive, both ensembles cannot be equivalent. It has been argued that the statistical formalism appropriate for astrophysical systems is given by the microcanonical ensemble [9].

Microcanonical thermodynamics has been recognized as especially useful for a statistical description of systems which suffer fragmentation and clustering [11], since there seems to be a problem in describing spatial inhomogeneities within the canonical ensemble. These phenomena, fragmentation and clustering, appear in many different branches of physics, from nuclear physics [13] and atomic clusters [14] to astrophysics [15]. The task of microcanonical thermodynamics is to compute the entropy of a given system as a function of the macroscopically (additive) conserved quantities. It is intuitively clear that the entropy of a system which undergoes a phase transition associated with spatial inhomogeneities, like clustering or collapsing, will depend crucially on the value of the total angular momentum. (See Ref. [16] for a similar discussion in a different context.) Indeed, angular momentum induces a repulsive centrifugal pseudopotential, which competes with the attractive potential, and which is able to modify the transition from the HP to the CP. Angular momentum is of particular importance in astrophysics:

the formation of a single star or a binary system, the birth of a solar system around a star, or the merger of galaxies in clusters are, to a large extent, determined by the value of the angular momentum.

A system of classical gravitating particles interacting via the Newton potential has an infinite entropy, due to the singularities at both short and long distances [9]. It is clear that the short distance singularity is not physical, since new physics—like quantum mechanics—appears at small scales; hence there should exist a natural short distance cutoff. The long distance singularity has a different nature. In principle, one is tempted to say that gravity should be able to bind the system. A little thought, however, easily convinces one that, at least from a statistical point of view, the system ''prefers'' to evaporate rather than to remain bound. A box (the long distance cutoff) is necessary to keep the system confined, but no such box appears in nature. We have to consider it as an artifact, making a statistical description possible. This is only sensible if the evaporation rate is small. The box breaks translational invariance, and therefore linear momentum is not conserved. Since we are interested in keeping the angular momentum conserved, we must deal with spherical boxes in order to maintain the rotational symmetry exactly.

The paper is organized as follows: in Sec. II we compute the microcanonical distribution with conserved angular momentum by integrating out the momenta in phase space. The resulting formula yields a microcanonical weight suitable for Monte Carlo simulations, and also allows an analytical approach based on mean field methods. In Sec. III we briefly derive the equations for the mean field approach. Section IV is devoted to a discussion of a simple model, introduced by Thirring [4], which mimics the main features of self-gravitating systems fairly well. The model is extended to take into account the conservation of angular momentum. The paper ends with a summary of the conclusions in Sec. V.

II. MICROCANONICAL DISTRIBUTION

Consider a system of N classical particles whose interactions are given by some potential energy Φ, depending only on the position of the particles. The Hamiltonian reads

$$\mathcal{H}=\sum_{i=1}^{N}\frac{\boldsymbol{p}_i^2}{2m_i}+\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N). \tag{2}$$

If the system is isolated, and the potential energy is translationally and rotationally invariant, the energy, momentum, and angular momentum will be conserved. As a consistency condition, the center of mass will move with a constant velocity. Without loss of generality, we can take the total linear momentum to be zero, and then the center of mass is fixed, for instance, at the origin of the coordinates. Assuming some kind of ergodicity, the statistical distribution will be flat on the available phase space shell. The entropy S is defined through the Boltzmann formula

$$S(E,L,N)=\ln W(E,L,N), \tag{3}$$

where E is the total energy and L the magnitude of the angular momentum. By symmetry, the entropy must be independent of the $\boldsymbol{L}$ direction. W is the volume of the phase space shell defined by the orbits with given E and $\boldsymbol{L}$. Notice that the Boltzmann constant is set equal to unity and the entropy is dimensionless; hence the temperature is measured in units of energy.

To avoid overly cumbersome expressions, let us consider the case in which only the angular momentum is conserved. If the linear momentum is also conserved, the derivation of the microcanonical distribution is similar, and the result will be reported at the end of this section. The volume of the relevant phase space shell can be computed starting from its definition:

$$W(E,L,N)=\frac{1}{N!}\int\left(\prod_{i=1}^{N}\frac{d^3r_i d^3p_i}{2\pi\hbar}\right)\delta(E-\mathcal{H})\,\delta^{(3)}\left(\boldsymbol{L}-\sum_i \boldsymbol{r}_i\times\boldsymbol{p}_i\right). \tag{4}$$

After integrating out the $\boldsymbol{p}$'s, we obtain a nonsingular microcanonical distribution depending only on the spatial configuration $\{\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\}$. In order to perform the integration over $\boldsymbol{p}$ in Eq. (4) easily, let us define

$$\Pi(\bar{E},L,N,\{\boldsymbol{r}\})=\int\prod_{i=1}^{N}d^3p_i\,\delta\left(\bar{E}-\sum_i\frac{p_i^2}{2m_i}\right)\delta^{(3)}\left(\boldsymbol{L}-\sum_i\boldsymbol{r}_i\times\boldsymbol{p}_i\right), \tag{5}$$

where we have used the notation $\bar{E}=E-\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$. It is convenient to take the Laplace transform of Π respect to $\bar{E}$,

$$\tilde{\Pi}(s,L,N,\{\boldsymbol{r}\})=\int_0^\infty d\bar{E}\,e^{-s\bar{E}}\Pi(\bar{E},L,N,\{\boldsymbol{r}\}), \tag{6}$$

where $\mathrm{Re}\,s>0$. Introducing the following representation for the remaining Dirac δ function,

$$\delta(x)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega x}, \tag{7}$$

we obtain

$$\tilde{\Pi}(s,L,N,\{\boldsymbol{r}\}) = \int \frac{d^3\omega}{(2\pi)^3} \exp(i\boldsymbol{\omega}\cdot\boldsymbol{L}) \int \prod_{i=1}^{N} d^3p_i \exp\left(-s\sum_i \frac{p_i^2}{2m_i} - i\sum_i \boldsymbol{\omega}\cdot(\boldsymbol{r}_i\times\boldsymbol{p}_i)\right). \tag{8}$$

The integral over $\boldsymbol{p}$ is now Gaussian, and can be readily performed. What remains is again a Gaussian integral over $\boldsymbol{\omega}$; as before, there is no difficulty in evaluating it. After some algebra, it is found that

$$\tilde{\Pi}(s,L,N,\{\boldsymbol{r}\}) = C(\det I)^{-1/2} \frac{e^{-s\,1/2\,L^T I^{-1} L}}{s^{(3N-3)/2}}, \tag{9}$$

where $L^T I^{-1} L = \Sigma_{\alpha\beta=1}^{3} L_\alpha I_{\alpha\beta}^{-1} L_\beta$, the matrix I is the moment of inertia tensor respect to the origin,

$$I_{\alpha\beta} = \sum_{i=1}^{N} m_i (r_i^2 \delta_{\alpha\beta} - r_i^\alpha r_i^\beta), \tag{10}$$

$\alpha,\beta = 1,2,3$ label the coordinates, and $C = (2\pi)^{(3N-3)/2} \Pi_i m_i^{3/2}$ is a constant.

The inverse Laplace transform of Eq. (9) can be found in any book of integral transform tables [17]. It is

$$\frac{e^{-sb}}{s^\nu} \longrightarrow \begin{cases} 0 & \text{if } 0<\bar{E}<b \\ \frac{1}{\Gamma(\nu)}(\bar{E}-b)^{\nu-1} & \text{if } \bar{E}>b. \end{cases} \tag{11}$$

Therefore,

$$\Pi(\bar{E},L,N,\{\boldsymbol{r}\}) = \frac{C}{\Gamma\left(\frac{3N-3}{2}\right)} (\det I)^{-1/2} \left(\bar{E} - \tfrac{1}{2} L^T I^{-1} L\right)^{(3N-5)/2} \tag{12}$$

if $\bar{E} > \frac{1}{2} L^T I^{-1} L$, and it vanishes otherwise.

After integrating out the $\boldsymbol{p}$'s, the volume of the phase space is given by

$$W(E,L,N) = \tilde{C} \int \left(\prod_{i=1}^{N} \frac{d^3 r_i}{2\pi\hbar}\right) \frac{1}{\sqrt{\det I}} \left(E - \tfrac{1}{2} L^T I^{-1} L - \Phi\right)^{(3N-5)/2}, \tag{13}$$

where $\tilde{C} = C/[N!\,\Gamma((3N-3)/2)]$. The domain of integration in Eq. (13) is restricted to the subset of the configuration space where the integrand is positive. The thermodynamical quantities can be computed numerically from the above integral by using a suitable Monte Carlo algorithm. Equation (13) can also be used to derive microcanonical equations in mean field approximations.

If the linear momentum is also conserved, a similar expression holds. Only the exponent in Eq. (13) must be replaced by $(3N-8)/2$, the constant C by

$$C = (2\pi)^{(3N-6)/2} \left(\sum_i m_i\right)^{-3/2} \prod_i m_i^{3/2}, \tag{14}$$

and the argument of the Γ function appearing in $\tilde{C}$ by $(3N-6)/2$. In this case, the moment of inertia tensor I refers to the center of mass, which can be taken at the origin, and then Eq. (10) holds. A Dirac δ function $\delta^{(3)}(\Sigma m_i \boldsymbol{r}_i)$ must be inserted into Eq. (13) in order to keep the center of mass fixed. Notice that, if the number of particles is large, momentum conservation gives negligible differences.

If the particles have some internal spin, the intrinsic moment of inertia of each particle must be added to the orbital moment of inertia, and the exponent on the right-hand side of Eq. (13) must be properly modified to take into account the number of intrinsic degrees of freedom. The derivation of a microcanonical weight for these more general cases is identical to that outlined in this section, and presents no additional difficulty. As a last remark, we stress that the results of this section, namely, Eq. (13), apply to stable as well as to unstable systems.

III. MEAN FIELD THEORY

If the number of particles N is very large, one can expect that the force a particle undergoes will be more sensitive to the mean particle distribution than to the fluctuations around it. Then one might think that the thermodynamics depends on the mean particle density alone. In this section, we shall give a derivation of the mean field equations using this hypothesis (cf. Ref. [18]). For reasons which will become clear in the following, the results of this section apply only to unstable systems.

Let us start with Eq. (13). To perform the integral over $\boldsymbol{r}_i$ for any i, we divide the integration region in Λ cells of volume a^3. The integral over d^3r_i becomes a sum over all possible cells. We have N such sums, one for each i. For each configuration $\{\boldsymbol{r}\}$ we can define a function $n(l)$, with $l=1,\ldots,\Lambda$, which counts the number of particles in the lth cell. It is not difficult to see that the initial sum over the position of the particles can be reorganized as a sum over the number of particles inside each cell, as follows:

$$W(E,L,N)=A\sum_{n(1)=0}^{N}\ \ldots\sum_{n(\Lambda)=0}^{N}\delta\left(\sum_{l=1}^{\Lambda}n(l)-N\right)\frac{N!}{\prod_{l'=1}^{\Lambda}n(l')!}\frac{1}{\sqrt{\det I}}\left(E-\tfrac{1}{2}L^TI^{-1}L-\Phi\right)^{(3N-5)/2}, \tag{15}$$

where A is a constant which contains $\tilde{C}$, the elementary phase space volume $2\pi\hbar$, and the volume of the cell a^3. The combinatorial factor is the number of configurations of particle positions which give the same occupation distribution $\{n(l)\}$. Introducing the particle density $\rho(\boldsymbol{r}_l)=n(l)/(Na^3)$, where $\boldsymbol{r}_l$ denotes the position of the center of the lth cell, we can write Eq. (15) as the following functional integral:

$$W(E,L,N)=\int[d\rho]\delta\left(\int d^3r\,\rho(\boldsymbol{r})-1\right)\exp\left\{N\left[-\int d^3r\,\rho(\boldsymbol{r})[\ln\rho(\boldsymbol{r})-1]+\tfrac{3}{2}\ln\left(E-\tfrac{1}{2}L^TI^{-1}L-\Phi\right)\right]\right\}, \tag{16}$$

where we have ignored some irrelevant constants and approximated the factorials by their asymptotic form, since we consider N large. We have also neglected the factor $\sqrt{\det I}$, which is of order $1/N$ with respect to the leading term. Now we have to clarify how the argument of the logarithm scales with N. The usual thermodynamical limit is reached by taking $N\to\infty$, keeping N/V,E/N, and L/N constant. The entropy per particle thus has a well defined limit, if the potential is stable. However, this limit does not exist for systems whose dynamics are governed by unstable potentials. Nevertheless, in these cases we can still take $N\to\infty$, by letting the masses and the coupling constants in the potential energy simultaneously go to zero in such a way that $Nm=M$ and $N^2\phi(\boldsymbol{r},\boldsymbol{r}')$ are kept constant. The energy E, the volume V, and the angular momentum L are not scaled. The thermodynamical functions depend on the total energy, instead of on the energy per particle. However, as can be seen from Eq. (16), the entropy scales with N. This scaling, which applies naturally to unstable systems, is the one we will consider in this paper. This is not a thermodynamical limit, but a continuous limit. It is sometimes called the Vlasov limit in the literature (see for instance [19]).

The functions I and Φ in Eq. (16) depend on $\rho(\boldsymbol{r}_l)$ through a suitable discretization. For very large N and very small a, they can be expressed as

$$I_{\alpha\beta}[\rho]=M\int d^3r\,\rho(\boldsymbol{r})(r^2\delta_{\alpha\beta}-r_\alpha r_\beta),$$

$$\Phi[\rho]=\int d^3r\,d^3r'\,\phi(\boldsymbol{r},\boldsymbol{r}')\rho(\boldsymbol{r})\rho(\boldsymbol{r}'). \tag{17}$$

The functional integral (16) is defined by the previous discretization. There are two limits to be considered in Eq. (16): $a\to0$ and $N\to\infty$. This is the rigorous order of the limits, since the discretization must be removed first in order to obtain the original $W(E,L,N)$. Afterwards, the number of particles can be set to infinity with the above prescription for the scaling. For unstable systems, mean field theory consists of interchanging both limits, first taking $N\to\infty$. In this case, the functional integral (16) is dominated by the maximum of the exponent of the integrand. All the fluctuations around the maximizing density are suppressed by powers of $1/N$. Therefore, all correlations are neglected and the mean field results are recovered. These are exact if the N and a limits commute. This offers an explanation for the well known fact that the physics of many systems with long range interactions is exactly described by mean field theory. For stable systems, the scaling in the thermodynamical limit is different, and the above arguments do not apply. Ignoring all the irrelevant constants, the entropy per particle can be defined as

$$\mathcal{S}=-\int d^3r\,\rho(\boldsymbol{r})[\ln\rho(\boldsymbol{r})-1]+\tfrac{3}{2}\ln\left(E-\tfrac{1}{2}L^TI^{-1}L-\Phi\right), \tag{18}$$

where I and Φ are given by Eq. (17). The equilibrium density ρ is that which maximizes $\mathcal{S}$ under the constraint $\int\rho(\boldsymbol{r})=1$.

The particle density which maximizes Eq. (18) must have a principal axis of the moment of inertia tensor along the angular momentum direction. Otherwise, it would be possible to define a new density by rotating the given one, in such a way that the principal axis with larger moment of inertia coincided with the angular momentum direction. The potential energy Φ and the pure entropical term $\int\rho(\ln\rho-1)$ would not change, but the rotational energy would be lowered, thus increasing $\mathcal{S}$. Taking the angular momentum along the z axis, we can write

$$\mathcal{S}=-\int d^3r\,\rho(\boldsymbol{r})[\ln\rho(\boldsymbol{r})-1]+\tfrac{3}{2}\ln\left(E-\frac{L^2}{2I_{33}}-\Phi\right). \tag{19}$$

The constrained maximum can be obtained from the functional derivative with respect to ρ. This gives the integral equation

$$\rho(\boldsymbol{r})=\exp\left\{\frac{3}{2}\beta\left[\xi(x^2+y^2)-\int d^3r'\,\phi(\boldsymbol{r},\boldsymbol{r}')\,\rho(\boldsymbol{r}')\right]+\mu\right\}, \tag{20}$$

where μ is the Lagrange multiplier for the constraint $\int\rho=1$, and we have used the notation

$$\beta=\left(E-\frac{L^2}{2I_{33}}-\Phi\right)^{-1}\quad\text{and}\quad\xi=\frac{ML^2}{2I_{33}^2}. \tag{21}$$

IV. THIRRING MODEL WITH ANGULAR MOMENTUM

Long ago, Thirring proposed a very simple model for a star [4]. In spite of its extreme simplicity, it mimics the main features of self-gravitating systems with surprisingly good success. (About the relevance of the Thirring model in astrophysics, and its comparison with more realistic models, see [9] and references therein.) Thirring considered the model without angular momentum. In this section, we shall see that, as expected, angular momentum substantially changes the behavior of the system.

A. Model

The model can be described as follows: a set of N particles is confined to a spherical volume V. Inside this volume there is a spherical interaction region (core) V_0, concentric with V. Particles outside the core (''atmosphere'') do not interact, and two particles inside the core have a constant attractive potential energy. Using the step function

$$\Theta_{V_0}(\boldsymbol{r})=\begin{cases}1 & \text{if } \boldsymbol{r}\in V_0\\ 0 & \text{if } \boldsymbol{r}\notin V_0,\end{cases} \tag{22}$$

the interaction energy can be written as

$$\phi(\boldsymbol{r},\boldsymbol{r}')=-\frac{Gm^2}{2}\Theta_{V_0}(\boldsymbol{r})\Theta_{V_0}(\boldsymbol{r}'), \tag{23}$$

where m is the mass of a particle and G the ''gravitational'' constant. (We have chosen the coupling constant in analogy with a self-gravitating system.) We can carry out the $N\to\infty$ limit of Sec. III, with $Nm=M$ fixed. Then the potential energy becomes

$$\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)=-\frac{GM^2}{2}\alpha^2, \tag{24}$$

where $\alpha=\int_{V_0}d^3r\,\rho(\boldsymbol{r})$ is the fraction of particles inside V_0.

B. Mean field equations

To simplify the computations, we shall consider the model in two dimensions. (It should not be difficult to solve it in three dimensions, but it is slightly more cumbersome, and no qualitative difference is expected.) In this case, the factor $\frac{3}{2}$ which appears in front of β in Eq. (20) must be replaced by 1. Let us introduce the following notation: R and R_0, respectively, are the radii of the total volume V and the core V_0, and $\kappa=V_0/V$. Using the dimensionless variables $\epsilon=E/(GM^2)$ and $\Omega=L^2/(2GM^3R^2)$, the mean field equation (20) becomes

$$\rho(\boldsymbol{r})=\begin{cases}\exp\{\mu+\beta\alpha+\beta\xi r^2/R^2\} & \text{if } r<R_0\\ \exp\{\mu+\beta\xi r^2/R^2\} & \text{if } r>R_0,\end{cases} \tag{25}$$

where

$$\beta=\left(\epsilon-\frac{\Omega}{\frac{1}{R^2}\int d^2r\,r^2\,\rho(\boldsymbol{r})}+\frac{\alpha^2}{2}\right)^{-1},$$

$$\xi=\frac{\Omega}{\left[\frac{1}{R^2}\int d^2r\,r^2\,\rho(\boldsymbol{r})\right]^2}. \tag{26}$$

The two self-consistency equations

$$\alpha=\int_{V_0}d^2r\,\rho(\boldsymbol{r}),\quad 1-\alpha=\int_{V\setminus V_0}d^2r\,\rho(\boldsymbol{r}) \tag{27}$$

determine α and μ. From them, it is straightforward to derive the following equation for α:

$$\ln\alpha-\ln(1-\alpha)-\beta\alpha=-\beta\xi(1-\kappa)+\ln\left(\frac{1-e^{-\beta\xi\kappa}}{1-e^{-\beta\xi(1-\kappa)}}\right). \tag{28}$$

Notice that β is positive by definition. It is possible to eliminate μ from (25), so that the particle distribution becomes

$$\rho(\boldsymbol{r})=\begin{cases}\dfrac{\alpha F_0}{V}\exp\left(-\beta\xi\dfrac{r^2}{R^2}\right) & \text{if } \boldsymbol{r}\in V_0\\[2ex] \dfrac{(1-\alpha)F_1}{V}\exp\left(-\beta\xi\dfrac{r^2}{R^2}\right) & \text{if } \boldsymbol{r}\notin V_0,\end{cases} \tag{29}$$

where

$$F_0=\frac{\beta\xi}{e^{\beta\xi\kappa}-1}\quad\text{and}\quad F_1=\frac{\beta\xi}{e^{\beta\xi}-e^{\beta\xi\kappa}}. \tag{30}$$

Equation (28) provides the complete solution for the system. Once we obtain α, for different values of the energy ϵ and the angular momentum Ω, we can compute the entropy $\mathcal{S}$ and the temperature T. Owing to the fact that the mass distribution maximizes $\mathcal{S}$, we have

$$\frac{1}{T}=\frac{\partial\mathcal{S}[\rho,\epsilon]}{\partial\epsilon}=\beta. \tag{31}$$

Notice that α is an order parameter for the collapsing phase transition, since it is the fraction of particles inside the core

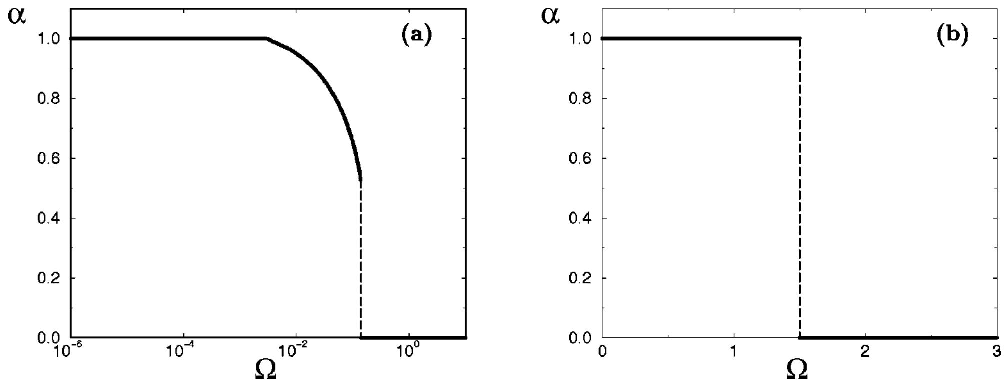

FIG. 1. The structure of the ground state as a function of Ω for (a) $\kappa<\frac{1}{2}$ (observe the logarithmic scale in abscissas) and (b) $\kappa>\frac{1}{2}$.

V_0. Notice also that Eq. (28) can have more than one solution for some values of ϵ and Ω. When this is the case, the true solution is that which maximizes the entropy. The others constitute metastable states [20].

C. Limiting cases

Let us study the solutions of Eq. (28) in some limiting cases.

(a) If $\Omega\rightarrow 0$, then $\xi\rightarrow 0$, and we recover the original Thirring model. There is a phase transition separating a low energy CP, where $\alpha\sim 1$, from a high energy HP, with $\alpha\sim\kappa$.

(b) If $\Omega\rightarrow\infty$, then $\xi\rightarrow\infty$. In this case $\alpha\rightarrow 0$ for $\epsilon\sim 4\Omega$ and $\alpha\rightarrow\kappa$ for $\epsilon\gg 4\Omega$. No collapse happens.

(c) For fixed values of Ω and $\epsilon\rightarrow\infty$, we have $\beta\rightarrow 0$, and then $\alpha\rightarrow\kappa$. In the high energy region, the system behaves as a gas, irrespective of the angular momentum, and the mass distribution is homogeneous.

(d) The most interesting case is the behavior of the ground state for fixed values of Ω. The ground state is the macroscopic state described by the microstates on the phase space shell with minimal energy for fixed angular momentum. It corresponds to zero temperature, since the energy of the system is saturated by the rotational energy (which cannot be zero due to the angular momentum) and the potential energy in such a way that no energy remains for thermal motion. This clearly implies $\beta\rightarrow\infty$. As can easily be seen, the entropy goes to $-\infty$ as $-\ln\beta$; hence, at zero temperature, the nonthermal (rotational plus potential) energy must be at its minimum. Otherwise, the system can change its mass distribution in such a way that some amount of thermal energy $1/\beta$ appears, increasing the entropy. In Sec. IV D, we shall see what the structure of the ground state is for different values of Ω.

D. Ground state

As we have just discussed, when $\beta\rightarrow\infty$, the nonthermal energy

$$\epsilon_{\rm nt}=\frac{\Omega}{\dfrac{1}{R^2}\displaystyle\int d^2r\, r^2\,\rho(\boldsymbol{r})}-\frac{\alpha^2}{2}\qquad(32)$$

must be at its minimum. It is possible to minimize the rotational energy without modifying the potential energy: for each fixed α, take a mass distribution which maximizes the moment of inertia. Obviously, this mass distribution corresponds to a fraction α of the particles in the outer layer of the core and the remaining $1-\alpha$ fraction in the outer layer of the system. The dimensionless moment of inertia is then $(1/R^2)\int d^2r\, r^2\,\rho(\boldsymbol{r})=1-\alpha(1-\kappa)$; hence the nonthermal energy is

$$\epsilon_{\rm nt}=\frac{\Omega}{1-\alpha(1-\kappa)}-\frac{\alpha^2}{2}.\qquad(33)$$

The absolute minimum of the above function for $\alpha\in[0,1]$ gives the equilibrium particle distribution in the ground state as a function of Ω and the parameter κ. Details are found in the Appendix, where it is shown that the behavior of the ground state depends on κ in the following way.

(a) $\kappa<\frac{1}{2}$ (atmosphere larger than the core). There are two critical values of the angular momentum: $\Omega_1=\kappa^2/(1-\kappa)$ and $\Omega_2=1/[8(1-\kappa)^2]$. For $\Omega<\Omega_1$ the ground state is characterized by a complete collapse with all the particles inside the core $(\alpha=1)$. For $\Omega_1<\Omega<\Omega_2$ there is an incomplete collapse with most, but not all, the particles in the core. The fraction of particles inside the core varies continuously from $\alpha=1$ at Ω_1 to $\alpha=1/[2(1-\kappa)]$ at Ω_2. For $\Omega>\Omega_2$ the ground state has an empty core $(\alpha=0)$. Notice the jump of α at Ω_2.

(b) $\kappa>\frac{1}{2}$ (atmosphere smaller than the core). There is one critical value of the angular momentum, $\Omega_3=\kappa/[2(1-\kappa)]$. The ground state for $\Omega<\Omega_3$ is characterized by a complete collapse with all the particles inside the core. For $\Omega>\Omega_3$ no collapse happens $(\alpha=0)$.

Figure 1 displays the behavior of the ground state as a function of Ω for the cases $\kappa<1/2$ and $\kappa>\frac{1}{2}$.

E. Phase diagram

For the discussion of the model at finite temperatures, we consider the most interesting case, $\kappa<1/2$ (atmosphere larger than the core). The numerical solutions displayed in the figures are for $\kappa=1/(e^3-1)\approx 0.0524$. Exactly as for the ground state, there are three different phases, which can be distinguished by the behavior of the order parameter α. Figure 2 displays α as a function of energy for three values of Ω corresponding to each phase. Figures 3 and 4 show the temperature and the entropy for the same values of Ω. The correlation between the collapsing transition and the anomalies

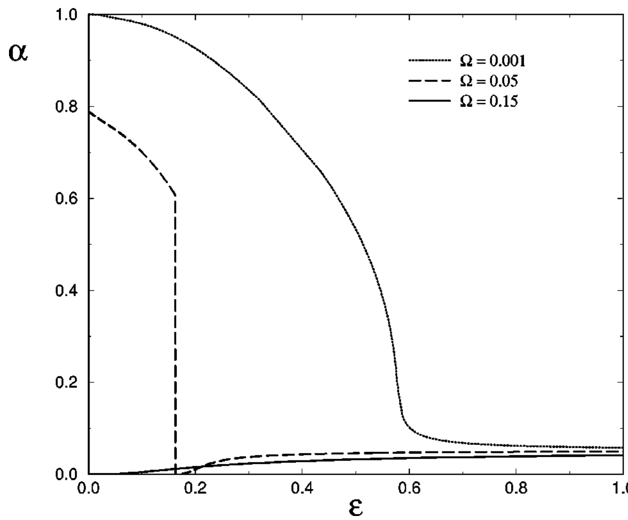

FIG. 2. The collapsing order parameter α vs the energy for Ω values corresponding to each of the three phases. To facilitate comparisons, we have shifted the energy such that the minimum energy is $\epsilon = 0$ for all Ω.

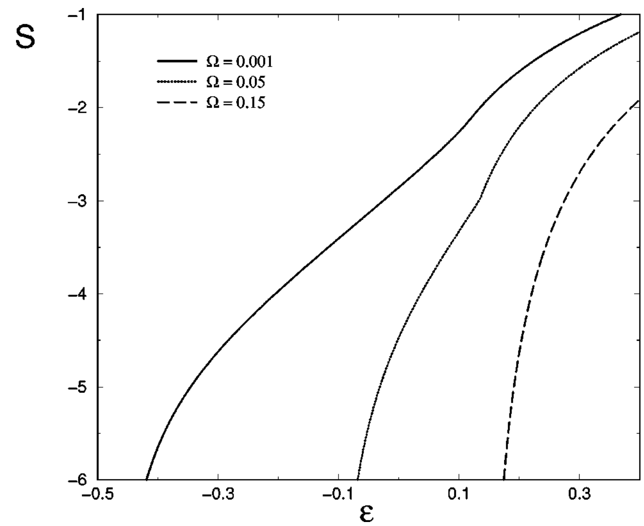

FIG. 4. The entropy vs the energy for three values of Ω, corresponding to each phase of the system. The convex intruder for the two smaller Ω can be seen. For $\Omega = 0.15$ the entropy is concave.

in the thermodynamical quantities is clearly seen. We now discuss in detail what is happening in each phase.

(1) $\Omega < \Omega_1$. There is a complete collapse at low energies, with $\alpha \approx 1$. The HP is separated from the CP by an interval of energies with negative specific heat. Qualitatively, the model is similar to the original ($\Omega = 0$) Thirring model. The entropy shows the characteristic convex intruder in the energy interval where the two phases coexist. (Notice that, since the system is not thermodynamically stable, van Hove's theorem [12] does not apply).

(2) $\Omega_1 < \Omega < \Omega_2$. At low energies the collapse is not complete, with $\alpha < 1$. For values of Ω near Ω_1 the thermodynamical quantities are qualitatively equal to those in the low angular momentum phase (negative specific heat). When Ω is larger, Eq. (28) has more than one solution in some energy interval. The solutions form three continuous branches. The stable state is determined by the solution with the largest entropy. The other solutions are metastable states [20]. There is a crossover at a certain energy: the stable solution moves from one branch to another. This is the origin of the jumps in α and in T that can be seen in Figs. 2 and 3 ($\Omega = 0.05$ in these plots). Notice that the jump occurs after a region with negative specific heat. For larger Ω, the jump appears before the specific heat becomes negative; however, even though in the last cases the specific heat is positive everywhere, the entropy has still a convex intruder, due to the kink originated by the jump in the temperature.

(3) $\Omega > \Omega_2$. There is no collapse at low energies. The specific heat is positive, smooth, and increases monotonically with the energy. The entropy has no convex intruder.

The above discussion demonstrates the correlation between the collapsing phase transition and the anomalies of the caloric curve (T versus ϵ), and in the entropy. The phase diagram in the plane (ϵ,Ω) is displayed in Fig. 5. Obviously, there is a forbidden region, since a minimal rotational energy is required to keep angular momentum constant. The bound-

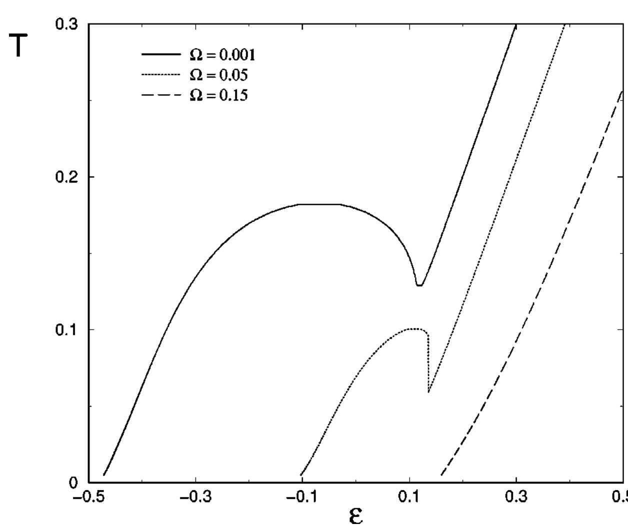

FIG. 3. The microcanonical temperature vs the energy for three values of Ω, corresponding to each phase of the system.

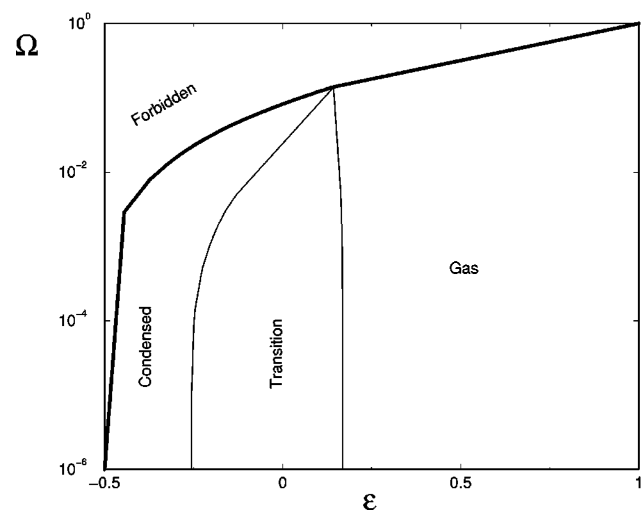

FIG. 5. The phase diagram of the model. Notice the logarithmic scale in the ordinate axis. The thick line corresponds to the $T = 0$ isotherm.

ary between the forbidden and allowed regions is the $T=0$ isotherm. Inside the allowed region, there are two phases separated by a transition region, which is defined through the Maxwell construction. (The two CP's, $\Omega<\Omega_1$ and $\Omega_1<\Omega<\Omega_2$, are distinguished only by the behavior of the order parameter at low energies. The thermodynamical quantities, however, behave in a similar way in both cases. Therefore, we do not distinguish the two low energy regimes in the phase diagram.) Since both phases are qualitatively different, they must be analytically separated. This means that the transition must end at zero temperature (i.e., on the curve of minimal energy). The transition region shrinks as Ω increases, and disappears at $\Omega_c=\Omega_2$ on the zero temperature line.

For fixed Ω, the difference of the energies limiting the transition region defines a ''latent heat.'' We see that the phase transition is first order everywhere, with non-vanishing latent heat. From the analysis of Eq. (28) when $\beta\to\infty$ and $\Omega\to\Omega_2$, one concludes that the latent heat vanishes as $\Omega_2-\Omega$. A careful study of the possible solutions, their entropies, etc., which is not reported here in detail, leads to this conclusion. Therefore, at the critical point Ω_2, the latent heat disappears. Since the specific heat is continuous at this point, we cannot say that the phase transition becomes second order; indeed, the order parameter α jumps at Ω_2.

F. Discussion

We now want to address the question of whether the phase diagram described in Sec. IV E shares its main properties with those of more realistic models, or whether these are merely a consequence of the extreme simplicity of the model. We can give at least a partial answer. The model considered here is unrealistic mainly because of the fact that the particles can only collapse in a fixed region. Therefore, there is no place for more complicated regimes like multifragmentation.

At low angular momentum, the CP, with all —or most— of the particles forming a cluster, will be present in realistic problems. Of course, the size of the cluster will increase with angular momentum, because of the centrifugal pseudopotential. For large energies, both the potential and rotational energy are negligible and we shall have a HP. There will then be a transition region separating the two phases, as in Fig. 5. However, in realistic cases, we do not expect the CP to disappear for large angular momentum. Instead, a CP phase with two or more clusters will appear. This is intuitively clear, since the ground state will contain one cluster for low angular momentum and two clusters if the system rotates faster. This property of the ground state with non-vanishing angular momentum can easily be verified in the Hertel-Thirring model [5], which is a generalization of the Thirring model that allows the particles to condense in different cells. The center of mass of the system must be held fixed, as is the case in isolated systems with negligible effects of the wall (self-gravitating systems, for instance). The size of the low energy region, where the CP appears, will shrink if angular momentum increases, and will certainly become zero when the system rotates infinitely fast. We expect a richer phase diagram, possibly with successive collapsing transitions, first into two (or more) clusters and afterwards into a single one. More complicated regimes, like multifragmentation, could also be present as metastable [20], or perhaps truly stable, states.

V. CONCLUSIONS

Let us briefly summarize the results presented in this work. First, we found a nonsingular expression for the microcanonical distribution of systems with conserved, nonvanishing angular momentum. It was obtained by integrating out the momenta in phase space, and it is suitable for Monte Carlo simulations as well as for mean field computations. We presented a derivation of the mean field equations for systems of classical particles interacting via a two body unstable potential, taking into account angular momentum conservation.

As an example, we discussed the properties of the phase diagram of a simple model of self-gravitating particles, including conservation of angular momentum. We found the exact solution in the mean field approach, which served to illustrate how the phase diagram of a physical system can be modified by the conservation of angular momentum. In this simple model, the angular momentum $L\propto\sqrt{\Omega}$ leads to the existence of three phases, separated by two critical values Ω_1 and Ω_2. At low energies, all particles condense in a small region if $\Omega<\Omega_1$. For $\Omega_1<\Omega<\Omega_2$, the collapse at low energy is incomplete, with most —but not all— particles condensed in a small region. At high energies, the system behaves like a gas, irrespective of the value of Ω. The transition from the HP to the CP is accompanied by an anomaly in the caloric curve (T versus ϵ), which reflects the fact that the entropy is not concave in an energy interval. This interval coincides with that where the homogeneous-collapse transition takes place. Through the Maxwell construction, it is possible to define this transition region and a latent heat, which shows that the transition can be classified as first order. The latent heat disappears continuously at $\Omega=\Omega_2$ and $T=0$. This reflects the fact that both phases, being qualitatively different, cannot be analytically connected in the phase diagram. At $\Omega>\Omega_2$ there is neither collapse nor anomaly in the thermodynamical quantities. The fact that the anomalies of the thermodynamical quantities disappear at large angular momentum, when the collapse disappears, unambiguously supports the close link between these two phenomena: collapse and thermodynamical anomalies.

We argued that, with more realistic potentials, the phase diagram will likely be richer. In particular, the CP will be present for large angular momentum, the collapse taking place into two clusters. There remain many open questions, for instance: do several transitions appear, with the system successively collapsing first into two or more clusters and subsequently into a single one? Is there any room for multifragmentation regimes, either as metastable or perhaps as stable states? It is the opinion of the author that all of these questions deserve an answer through extensive studies.

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APPENDIX

In this appendix we study in detail the minima of the nonthermal energy (see Sec. IV D)

$$\epsilon_{nt}=\frac{\Omega}{1-\alpha(1-\kappa)}-\frac{\alpha^2}{2}. \tag{A1}$$

The absolute minimum of the above function in $[0,1]$ is either at $\alpha=0,\alpha=1$, or at one solution of

$$\frac{\Omega(1-\kappa)}{[1-\alpha(1-\kappa)]^2}-\alpha=0. \tag{A2}$$

Equation (A2) is equivalent to a real polynomial equation of third degree, and therefore has either one or three real solutions. Since the first term in Eq. (A2) has a pole at $\alpha=1/(1-\kappa)>1$, there is always a solution at $\alpha>1/(1-\kappa)$, which is nonphysical since α must belong to $[0,1]$. Besides this nonphysical solution, Eq. (A2) has two more solutions when Ω is small. For $\Omega\ll 1$,

$$\alpha_{max}\approx\Omega(1-\kappa), \quad \alpha_{min}\approx\frac{1}{1-\kappa}(1-\sqrt{\Omega(1-\kappa)}). \tag{A3}$$

From the second derivative of Eq. (A2), we see that there is an inflection point at

$$\alpha_{ip}=\frac{1-\sqrt[3]{\Omega(1-\kappa)^2}}{1-\kappa}, \tag{A4}$$

so that α_{max} corresponds to a local maximum, and α_{min} to a local minimum. We always have $\alpha_{max}\leqslant\alpha_{ip}\leqslant\alpha_{min}$. Taking the derivative of Eq. (A2) with respect to Ω, with $\alpha=\alpha_{min}$, we easily see that α_{min} is a monotonically decreasing function of Ω. Analogously, introducing $\alpha=\alpha$min in Eq. (A1), taking the derivative respect to Ω and using Eq. (A2), we see that the value of ϵ_{nt} at its local minimum α_{min} is a monotonically increasing function of Ω.

Let us study the absolute minimum of Eq. (A1) for $0\leqslant\alpha\leqslant 1$. The three candidates are $\alpha=0,1$, or α_{min}. For small Ω, from Eq (A3) we have $\alpha_{min}>1$, and, since $\epsilon_{nt}(0)=\Omega$ and $\epsilon_{nt}(1)=\Omega/\kappa-1/2$, the absolute minimum corresponds to $\alpha=1$; however, as Ω grows, $\epsilon_{nt}(1)$ increases faster than $\epsilon_{nt}(0)$, and α_{min} will reach the physical region, since it decreases. The following three possibilities define three critical values of Ω.

(i) The relative minimum α_{min} reaches the physical region. The critical value Ω_1 is given by $\alpha_{min}(\Omega_1)=1$. From Eq. (A2), we obtain $\Omega_1=\kappa^2/(1-\kappa)$.

(ii) The relative minimum α_{min} ceases to be the absolute minimum. This happens for $\Omega=\Omega_2$, when $\epsilon_{nt}(\alpha_{min})=\epsilon_{nt}(0)$. We have the two equations

$$\alpha_{min}=\frac{\Omega_2(1-\kappa)}{[1-\alpha_{min}(1-\kappa)]^2}, \quad \Omega_2=\frac{\Omega_2}{1-\alpha_{min}}-\frac{\alpha_{min}^2}{2}.$$

The solution is $\Omega_2=1/[8(1-\kappa)^2]$, and $\alpha_{min}=1/[2(1-\kappa)]$.

(iii) The value of ϵ_{nt} at $\alpha=0$ surpasses that at $\alpha=1$. This happens when $\Omega=\Omega_3$, having $\epsilon_{nt}(0)=\epsilon_{nt}(1)$. Clearly, $\Omega_3=\kappa/[2(1-\kappa)]$.

The ground state behavior depends on the ordering of these critical Ω's, which in turn depends on κ. For $\kappa<\frac{1}{2}$ we have $\Omega_1<\Omega_2,\Omega_3$. The ground state is then characterized by complete collapse if $\Omega<\Omega_1$, with all the particles inside the core. For $\Omega_1<\Omega<\Omega_2$ there is an incomplete, with most — but not all — of the particle in the core. The value of α varies continuously from $\alpha=1$ at Ω_1 to $\alpha=1/[2(1-\kappa)]$ at Ω_2. For $\Omega>\Omega_2$ the ground state has an empty core ($\alpha=0$). Notice the jump of α at Ω_2.

For $\kappa>\frac{1}{2}$ we have $\Omega_3<\Omega_1$ and $\Omega_3<\Omega_2$. In this case, the ground state for $\Omega<\Omega_3$ is characterized by a complete collapse, with all the particles inside the core. For $\Omega>\Omega_3$ no collapse happens ($\alpha=0$).

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